

how two ex-students turned on to pure
mathematics and found total happiness



a mathematical novelette by D. E. Knuth

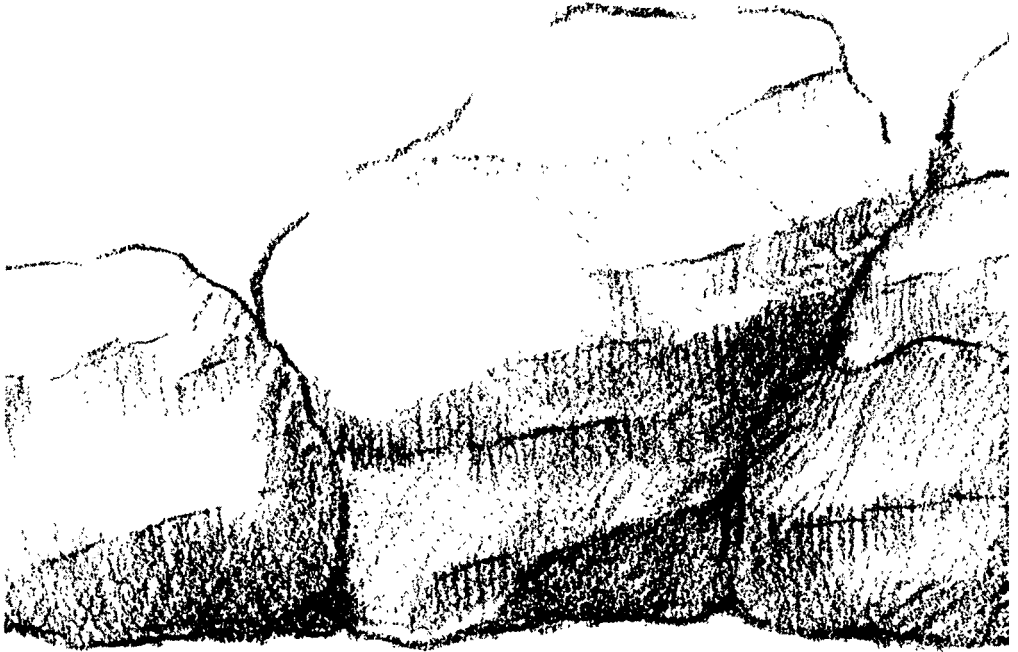
SURREAL NUMBERS

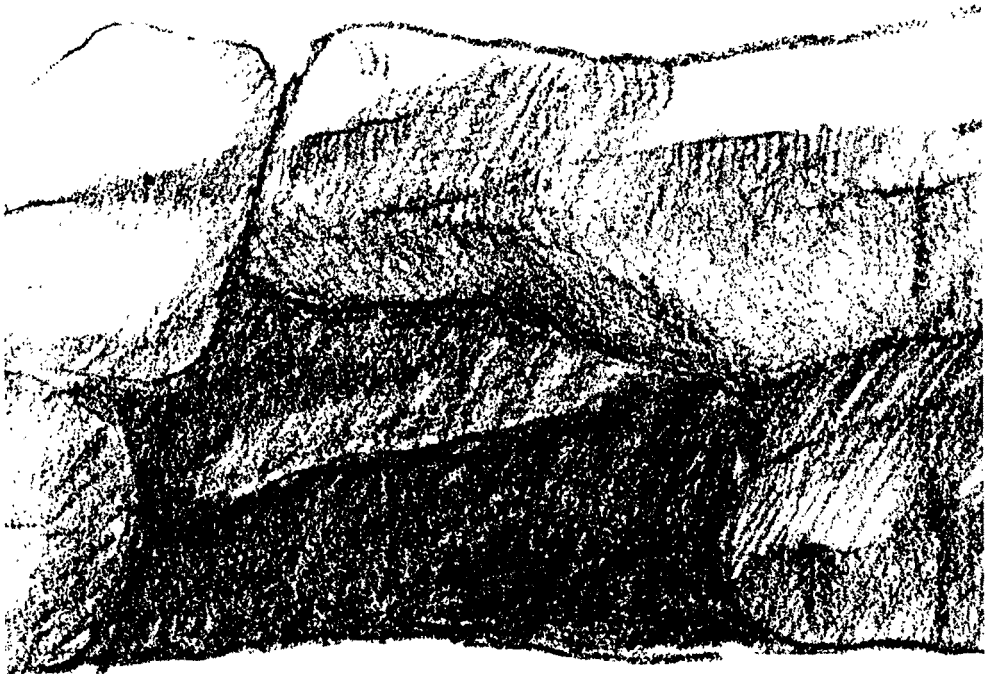


ADDISON WESLEY

1

THE ROCK





A. Bill, do you think you've found yourself?

B. What?

A. I mean — here we are on the edge of the Indian Ocean, miles away from civilization. It's been months since we ran off to avoid getting swept up in the system, and "to find ourselves." I'm just wondering if you think we've done it.

B. Actually, Alice, I've been thinking about the same thing. These past months together have been really great — we're completely free, we know each other, and we feel like real people again instead of like machines. But lately I'm afraid I've been missing some of the things we've "escaped" from. You know, I've got this fantastic craving for a book to read — *any* book, even a textbook, even a math textbook. It sounds crazy, but I've been lying here wishing I had a crossword puzzle to work on.

A. Oh, c'mon, not a crossword puzzle; that's what your *parents* like to do. But I know what you mean, we need some mental stimulation. It's kinda like the end of summer vacations when we were kids. In May every year we couldn't wait to get out of school, and the days simply dragged on until vacation started, but by September we were real glad to be back in the classroom.

B. Of course, with a loaf of bread, a jug of wine, and thou beside me, these days aren't exactly "dragging on." But I think maybe the most important thing I've learned on this trip is that the simple, romantic life isn't enough for me. I need something complicated to think about.

A. Well, I'm sorry I'm not complicated enough for you. Why don't we get up and explore some more of the beach? Maybe we'll find some pebbles or something that we can use to make up some kind of a game.

B. (sitting up) Yeah, that's a good idea. But first I think I'll take a little swim.

A. (running toward the water) Me, too — bet you can't catch me!

.

B. Hey, what's that big black rock half-buried in the sand over there?

- A. Search me, I've never seen anything like it before. Look, it's got some kind of graffiti on the back.
- B. Let's see. Can you help me dig it out? It looks like a museum piece. Unnh! Heavy, too. The carving might be some old Arabian script . . . no, wait, I think it's maybe Hebrew; let's turn it around this way.
- A. Hebrew! Are you sure?
- B. Well, I learned a lot of Hebrew when I was younger, and I can almost read this. . . .
- A. I heard there hasn't been much archæological digging around these parts. Maybe we've found another Rosetta Stone or something. What does it say, can you make anything out?
- B. Wait a minute, gimme a chance. . . . Up here at the top right is where it starts, something like "In the beginning everything was void, and . . ."
- A. Far out! That sounds like the first book of Moses, in the Bible. Wasn't he wandering around Arabia for forty years with his followers before going up to Israel? You don't suppose . . .
- B. No, no, it goes on much different from the traditional account. Let's lug this thing back to our camp, I think I can work out a translation.
- A. Bill, this is wild, just what you needed!
- B. Yeah, I did say I was dying for something to read, didn't I. Although this wasn't exactly what I had in mind! I can hardly wait to get a good look at it — some of the things are kinda strange, and I can't figure out whether it's a story or what. There's something about numbers, and . . .
- A. It seems to be broken off at the bottom; the stone was originally longer.

- B. A good thing, or we'd never be able to carry it. Of course it'll be just our luck to find out the message is getting interesting, right when we come to the broken place.
- A. Here we are. I'll go pick some dates and fruit for supper while you work out the translation. Too bad languages aren't my thing, or I'd try to help you.
-
- B. Okay, Alice, I've *got* it. There are a few doubtful places, a couple signs I don't recognize; you know, maybe some obsolete word forms. Overall I think I know what it says, though I don't know what it means. Here's a fairly literal translation:

In the beginning, everything was void, and J. H. W. H. Conway began to create numbers. Conway said, "Let there be two rules which bring forth all numbers large and small. This shall be the first rule: Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set. And the second rule shall be this: One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number." And Conway examined these two rules he had made, and behold! They were very good. And the first number was created from the void left set and the void right set. Conway called this number "zero," and said that it shall be a sign to separate positive numbers from negative numbers. Conway proved that zero was less than or equal to zero, and he saw that it was good. And the evening and the morning were the day of zero. On the next day, two more numbers were created, one

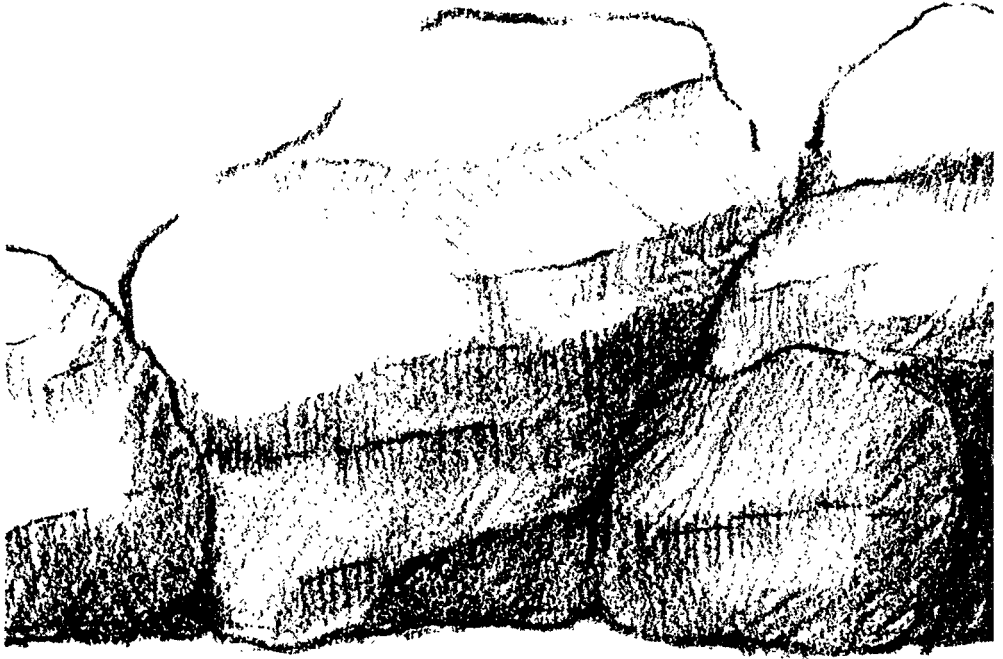
with zero as its left set and one with zero as its right set. And Conway called the former number “one,” and the latter he called “minus one.” And he proved that minus one is less than but not equal to zero and zero is less than but not equal to one. And the evening ...

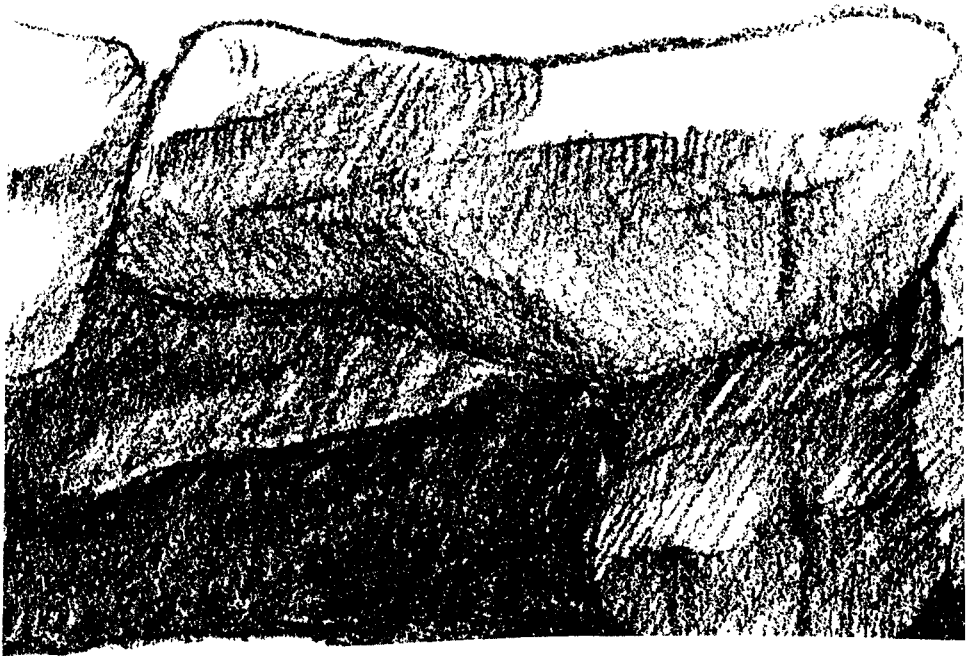
That’s where it breaks off.

- A. Are you *sure* it reads like that?
- B. More or less. I dressed it up a bit.
- A. But “Conway” ... that’s not a Hebrew name. You’ve got to be kidding.
- B. No, honest. Of course the old Hebrew writing doesn’t show any vowels, so the real name might be Keenawu or something; maybe related to the Khans? I guess not. Since I’m translating into English, I just used an English name. Look, here are the places where it shows up on the stone. The J. H. W. H. might also stand for “Jehovah.”
- A. No vowels, eh? So it’s real. ... But what do you think it means?
- B. Your guess is as good as mine. These two crazy rules for numbers. Maybe it’s some ancient method of arithmetic that’s been obsolete since the wheel was invented. It might be fun to figure them out, tomorrow; but the sun’s going down pretty soon so we’d better eat and turn in.
- A. Okay, but read it to me once more. I want to think it over, and the first time I didn’t believe you were serious.
- B. (pointing) “In the beginning, ...”

2

SYMBOLS





- A. I think your Conway Stone makes sense after all, Bill. I was thinking about it during the night.
- B. So was I, but I dozed off before getting anywhere. What's the secret?
- A. It's not so hard, really; the trouble is that it's all expressed in words. The same thing can be expressed in symbols and then you can see what's happening.

- B. You mean we're actually going to use the New Math to decipher this old stone tablet.
- A. I hate to admit it, but that's what it looks like. Here, the first rule says that every number x is really a pair of sets, called the left set x_L and the right set x_R :

$$x = (x_L, x_R).$$

- B. Wait a sec, you don't have to draw in the sand, I think we still have a pencil and some paper in my backpack. Just a minute. ... Here, use this.
- A. $x = (x_L, x_R)$.
 These x_L and x_R are not just numbers, they're *sets* of numbers; and each number in the set is itself a pair of sets, and so on.
- B. Hold it, your notation mixes me up. I don't know what's a set and what's a number.
- A. Okay, I'll use capital letters for sets of numbers and small letters for numbers. Conway's first rule is that

$$x = (X_L, X_R), \quad \text{where} \quad X_L \not\geq X_R. \quad (1)$$

This means if x_L is any number in X_L and if x_R is any number in X_R , they must satisfy $x_L \not\geq x_R$. And that means x_L is not greater than or equal to x_R .

- B. (scratching his head) I'm afraid you're still going too fast for me. Remember, you've already got this thing psyched out, but I'm still at the beginning. If a number is a pair of sets of numbers, each of which is a pair of sets of numbers, and so on and so on, how does the whole thing get started in the first place?
- A. Good point, but that's the whole beauty of Conway's scheme. Each element of X_L and X_R must have been created previously, but on the first day of creation there weren't any previous

number to work with; so both X_L and X_R were taken to be the empty set!

- B. I never thought I'd live to see the day when the empty set was meaningful. That's really creating something out of nothing, eh? But is $X_L \not\subseteq X_R$ when X_L and X_R are both equal to the empty set? How can you have something unequal itself?

Oh yeah, yeah, that's okay since it means no *element* of the empty set is greater than or equal to any element of the empty set — it's a true statement because there *aren't* any elements in the empty set.

- A. So everything gets started all right, and that's the number called zero. Using the symbol \emptyset to stand for the empty set, we can write

$$0 = (\emptyset, \emptyset).$$

- B. Incredible.

- A. Now on the second day, it's possible to use 0 in the left or right set, so Conway gets two more numbers

$$-1 = (\emptyset, \{0\}) \quad \text{and} \quad 1 = (\{0\}, \emptyset).$$

- B. Let me see, does this check out? For -1 to be a number, it has to be true that no element of the empty set is greater than or equal to 0. And for 1, it must be that 0 is not greater than any element of the empty set. Man, that empty set sure gets around! Someday I think I'll write a book called *Properties of the Empty Set*.

- A. You'd never finish.

If X_L or X_R is empty, the condition $X_L \not\subseteq X_R$ is true no matter *what* is in the other set. This means that infinitely many numbers are going to be created.

B. Okay, but what about Conway's second rule?

A. That's what you use to tell whether $X_L \not\leq X_R$, when both sets are nonempty; it's the rule defining less-than-or-equal. Symbolically,

$$x \leq y \quad \text{means} \quad X_L \not\leq y \quad \text{and} \quad x \not\leq Y_R. \quad (2)$$

B. Wait a minute, you're way ahead of me again. Look, X_L is a set of numbers, and y is a number, which means a pair of sets of numbers. What do you mean when you write " $X_L \not\leq y$ "?

A. I mean that every element of X_L satisfies $x_L \not\leq y$. In other words, no element of X_L is greater than or equal to y .

B. Oh, I see, and your rule (2) says also that x is not greater than or equal to any element of Y_R . Let me check that with the text.

...

A. The Stone's version is a little different, but $x \leq y$ must mean the same thing as $y \geq x$.

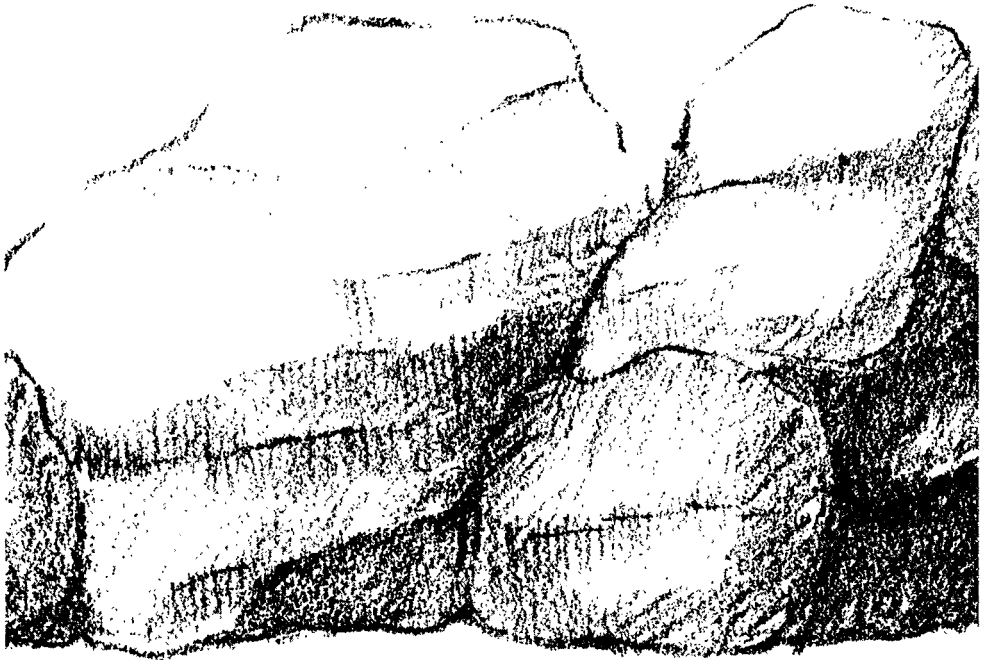
B. Yeah, you're right. Hey, wait a sec, look here at these carvings off to the side:

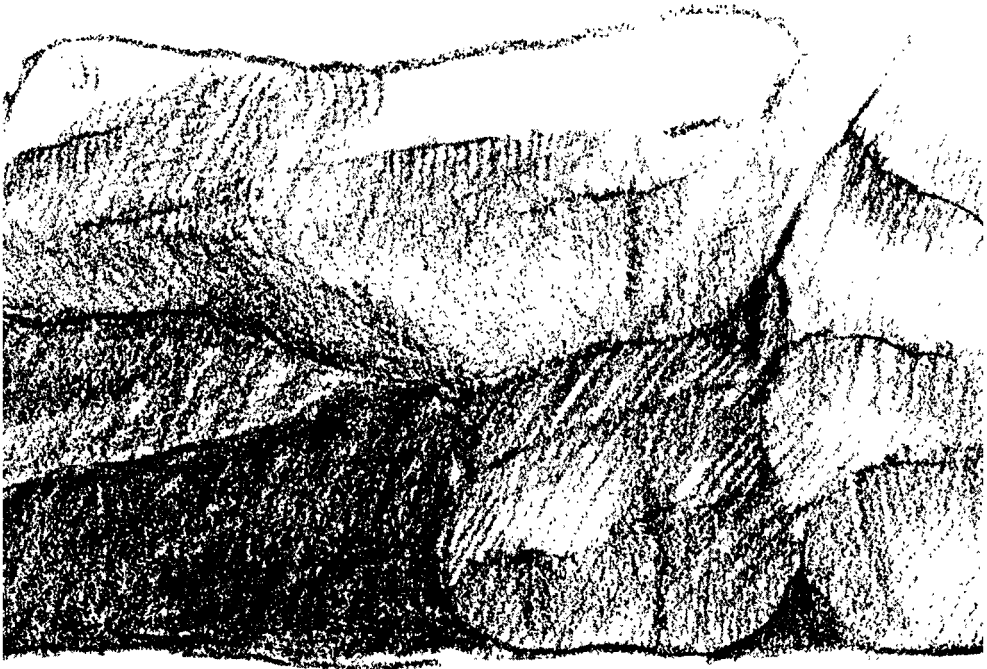
$$\begin{aligned} \bullet &= \langle \vdots \rangle \\ | &= \langle \bullet \vdots \rangle \\ - &= \langle \vdots \bullet \rangle \end{aligned}$$

These are the symbols I couldn't decipher yesterday, and your notation makes it all crystal clear! Those double dots separate the left set from the right set. You must be on the right track.

A. Wow, equal signs and everything! That stone-age carver must have used $-$ to stand for -1 ; I almost like his notation better than mine.

- B. I bet we've underestimated primitive people. They must have had complex lives and a need for mental gymnastics, just like us—at least when they didn't have to fight for food and shelter. We always oversimplify history when we look back.
- A. Yes, but otherwise how could we look back?
- B. I see your point.
- A. Now comes the part of the text I don't understand. On the first day of creation, Conway "proves" that $0 \leq 0$. Why should he bother to prove that something is less than or equal to itself, since it's obviously equal to itself. And then on the second day he "proves" that -1 is not equal to 0 ; isn't that obvious without proof, since -1 is a different number?
- B. Hmmm. I don't know about you, but I'm ready for another swim.
- A. Good idea. That surf looks good, and I'm not used to so much concentration. Let's go!





- B. An idea hit me while we were paddling around out there.
Maybe my translation *isn't* correct.
- A. What? It *must* be okay, we've already checked so much of
it out.
- B. I know; but now that I think of it, I wasn't quite sure of
that word I translated "equal to." Maybe it has a weaker
meaning, "similar to" or "like." Then Conway's second rule

becomes “One number is less than or *like* another number if and only if” And later on, he proves that zero is less than or *like* zero; minus one is less than but not like zero; and so forth.

- A. Oh, right, that must be it, he’s using the word in an abstract technical sense that must be defined by the rules. So of *course* he wants to prove that 0 is less than or like 0, in order to see that his definition makes a number “like” itself.
- B. So does his proof go through? By rule (2), he must show that no element of the empty set is greater than or like 0, and that 0 is not greater than or like any element of the empty set. . . . Okay, it works, the empty set triumphs again.
- A. More interesting is how he could prove that -1 is *not* like 0. The only way I can think of is that he proved that 0 is not less-than-or-like -1 . I mean, we have rule (2) to tell whether one number is less than or like another; and if x is not less-than-or-like y , it isn’t less than y and it isn’t like y .
- B. I see, we want to show that $0 \leq -1$ is false. This is rule (2) with $x = 0$ and $Y_R = \{0\}$, so $0 \leq -1$ if and only if $0 \not\leq 0$. But 0 *is* ≥ 0 , we know that, so $0 \not\leq -1$. He was right.
- A. I wonder if Conway also tested -1 against 1; I suppose he did, although the rock doesn’t say anything about it. If the rules are any good, there should be a way to prove that -1 is less than 1.
- B. Well, let’s see: -1 is $(\emptyset, \{0\})$ and 1 is $(\{0\}, \emptyset)$, so once again the empty set makes $-1 \leq 1$ by rule (2). On the other hand, $1 \leq -1$ is the same as saying that $0 \not\leq -1$ and $1 \not\leq 0$, according to rule (2), but we know that both of these are false. Therefore $1 \not\leq -1$, and it must be that $-1 < 1$. Conway’s rules seem to be working.
- A. Yes, but so far we’ve been using the empty set in almost every argument, so the full implications of the rules aren’t clear yet.

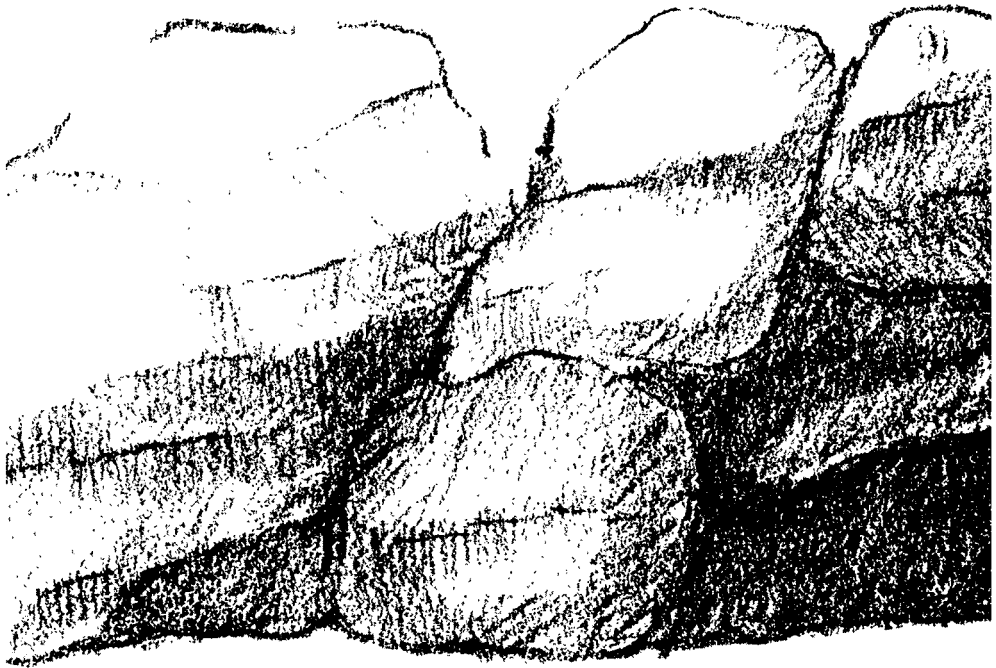
Have you noticed that almost everything we've proved so far can be put into a framework like this: "If X and Y are any sets of numbers, then $x = (\emptyset, X)$ and $y = (Y, \emptyset)$ are numbers, and $x \leq y$."

- B. It's neat the way you've just proved infinitely many things, by looking at the pattern I used in only a couple of cases. I guess that's what they call abstraction, or generalization, or something. But can you also prove that your x is strictly *less* than y ? That was true in all the simple cases and I bet it's true in general.
- A. Uh huh . . . Well no, not when X and Y are both empty, since that would mean $0 \not\leq 0$. But otherwise it looks very interesting. Let's look at the case when X is the empty set and Y is not empty; is it true that 0 is less than (Y, \emptyset) ?
- B. If so, then I'd call (Y, \emptyset) a "positive" number. That must be what Conway meant by zero separating the positive and negative numbers.
- A. Yes, but look. According to rule (2), we will have $(Y, \emptyset) \leq 0$ if and only if no member of Y is greater than or like 0 . So if, for example, Y is the set $\{-1\}$, then $(Y, \emptyset) \leq 0$. Do you want positive numbers to be ≤ 0 ?
- Too bad I didn't take you up on that bet.
- B. Hmm. You mean (Y, \emptyset) is going to be positive only when Y contains some number that is zero or more. I suppose you're right. But at least we now understand everything that's on the stone.
- A. Everything up to where it's broken off.
- B. You mean . . . ?
- A. I wonder what happened on the *third* day.

- B. Yes, we should be able to figure that out, now that we know the rules. It might be fun to work out the third day, after lunch.
- A. You'd better go catch some fish; our supply of dried meat is getting kinda low. I'll go try and find some coconuts.

4

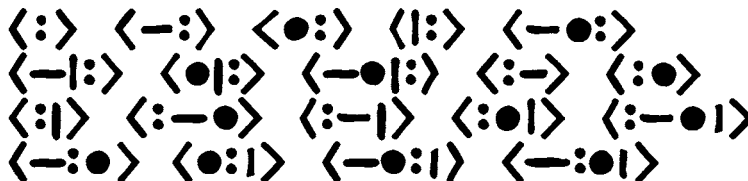
BAD NUMBERS





- B. I've been working on that Third Day problem, and I'm afraid it's going to be pretty hard. When more and more numbers have been created, the number of possible sets goes up fast. I bet that by the seventh day, Conway was ready for a rest.
- A. Right. I've been working on it too and I get seventeen numbers on the third day.

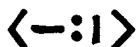
B. Really? I found nineteen; you must have missed two. Here's my list:



A. I see you're using the Stone's notation. But why did you include $\langle : \rangle$? That was created already on the first day.

B. Well, we have to test the new numbers against the old, in order to see how they fit in.

A. But I only considered *new* numbers in my list of seventeen, so there must actually be *twenty* different at the end of the third day. Look, you forgot to include



in your list.

B. (blinking) So I did. Hmm ... 20 by 20, that's 400 different cases we'll have to consider in rule (2). A lot of work, and not much fun either. But there's nothing else to do, and I know it'll bug me until I know the answer.

A. Maybe we'll think of some way to simplify the job once we get started.

B. Yeah, that would be nice. ...

Well, I've got one result, 1 is less than $(\{1\}, \emptyset)$. First I had to prove that $0 \not\geq (\{1\}, \emptyset)$.

A. I've been trying a different approach. Rule (2) says we have to test every element of X_L to see that it isn't greater than or like y , but it shouldn't be necessary to make so many tests. If

any element of X_L is $\geq y$, then the *largest* element of X_L ought to be $\geq y$. Similarly, we need only test x against the *smallest* element of Y_R .

- B. Yeah, that oughta be right. . . . I can prove that 1 is less than $(\{0, 1\}, \emptyset)$ just like I proved it was less than $(\{1\}, \emptyset)$; the extra “0” in X_L didn’t seem to make any difference.
- A. If what I said is true, it will save us a lot of work, because each number (X_L, X_R) will behave in all \leq relations exactly as if X_L were replaced by its largest element and X_R by its smallest. We won’t have to consider any numbers in which X_L or X_R have two or more elements; ten of those twenty numbers in the list will be eliminated!
- B. I’m not sure I follow you, but how on earth can we prove such a thing?
- A. What we seem to need is something like this:

$$\text{if } x \leq y \quad \text{and} \quad y \leq z, \quad \text{then} \quad x \leq z. \quad (\text{T1})$$

I don’t see that this follows immediately, although it is consistent with everything we know.

- B. At any rate, it ought to be true, if Conway’s numbers are to be at all decent. We could go ahead and assume it, but it would be neat to show once and for all that it is true, just by using Conway’s rules.
- A. Yes, and we’d be able to solve the Third Day puzzle without much more work. Let’s see, how can it be proved? . . .
- B. Blast those flies! Just when I’m trying to concentrate. Alice, can you — no, I guess I’ll go for a little walk.

.

Any progress?

- A. No, I seem to be going in circles, and the \nless versus \leq is confusing. Everything is stated negatively and things get incredibly tangled up.
- B. Maybe (T1) isn't true.
- A. But it *has* to be true. Wait, that's it! We'll try to *disprove* it. And when we fail, the cause of our failure will be a proof!
- B. Sounds good — it's always easier to prove something wrong than to prove it right.
- A. Suppose we've got three numbers x , y , and z for which

$$x \leq y, \quad \text{and} \quad y \leq z, \quad \text{and} \quad x \nless z.$$

What does rule (2) tell us about “bad numbers” like this?

- B. It says that

$$\begin{aligned} X_L \nless y, \\ \text{and} \quad x \nless Y_R, \\ \text{and} \quad Y_L \nless z, \\ \text{and} \quad y \nless Z_R, \end{aligned}$$

and then also $x \nless z$, which means what?

- A. It means one of the two conditions fails. Either there is a number x_L in X_L for which $x_L \geq z$, or there is a number z_R in Z_R for which $x \geq z_R$. With all these facts about x , y , and z , we ought to be able to prove *something*.
- B. Well, since x_L is in X_L , it can't be greater than or like y . Say it's less than y . But $y \leq z$, so x_L must be ... no, sorry, I can't use facts about numbers we haven't proved.

Going the other way, we know that $y \leq z$ and $z \leq x_L$ and $y \nless x_L$; so this gives us three more bad numbers, and we can get more facts again. But that looks hopelessly complicated.

- A. Bill! You've got it.
- B. Have I?
- A. If (x, y, z) are three bad numbers, there are two possible cases.
Case 1, some $x_L \geq z$: Then (y, z, x_L) are three more bad numbers.
Case 2, some $z_R \leq x$: Then (z_R, x, y) are three more bad numbers.
- B. But aren't you still going in circles? There's more and more bad numbers all over the place.
- A. No, in each case the new bad numbers are *simpler* than the original ones; one of them was created earlier. We can't go on and on finding earlier and earlier sets of bad numbers, so there can't be any bad sets at all!
- B. (brightening) Oho! What you're saying is this: Each number x was created on some day $d(x)$. If there are three bad numbers (x, y, z) , for which the sum of their creation days is $d(x) + d(y) + d(z) = n$, then one of your two cases applies and gives three bad numbers whose day-sum is less than n . Those, in turn, will produce a set whose day-sum is still less, and so on; but that's impossible since there are no three numbers whose day-sum is less than 3.
- A. Right, the sum of the creation days is a nice way to express the proof. If there are no three bad numbers (x, y, z) whose day-sum is less than n , the two cases show that there are none whose day-sum equals n . I guess it's a proof by induction on the day-sum.
- B. You and your fancy words. It's the *idea* that counts.
- A. True; but we need a name for the idea, so we can apply it more easily next time.
- B. Yes, I suppose there will be a next time. ...

Okay, I guess there's no reason for me to be uptight any more about the New Math jargon. You know it and I know it, we've just proved the *transitive law*.

- A. (sigh) Not bad for two amateur mathematicians!
- B. It was really your doing. I hereby proclaim that the transitive law (T1) shall be known henceforth as Alice's Theorem.
- A. C'mon. I'm sure Conway discovered it long ago.
- B. But does that make your efforts any less creative? I bet every great mathematician started by rediscovering a bunch of "well known" results.
- A. Gosh, let's not get carried away dreaming about greatness! Let's just have fun with this.